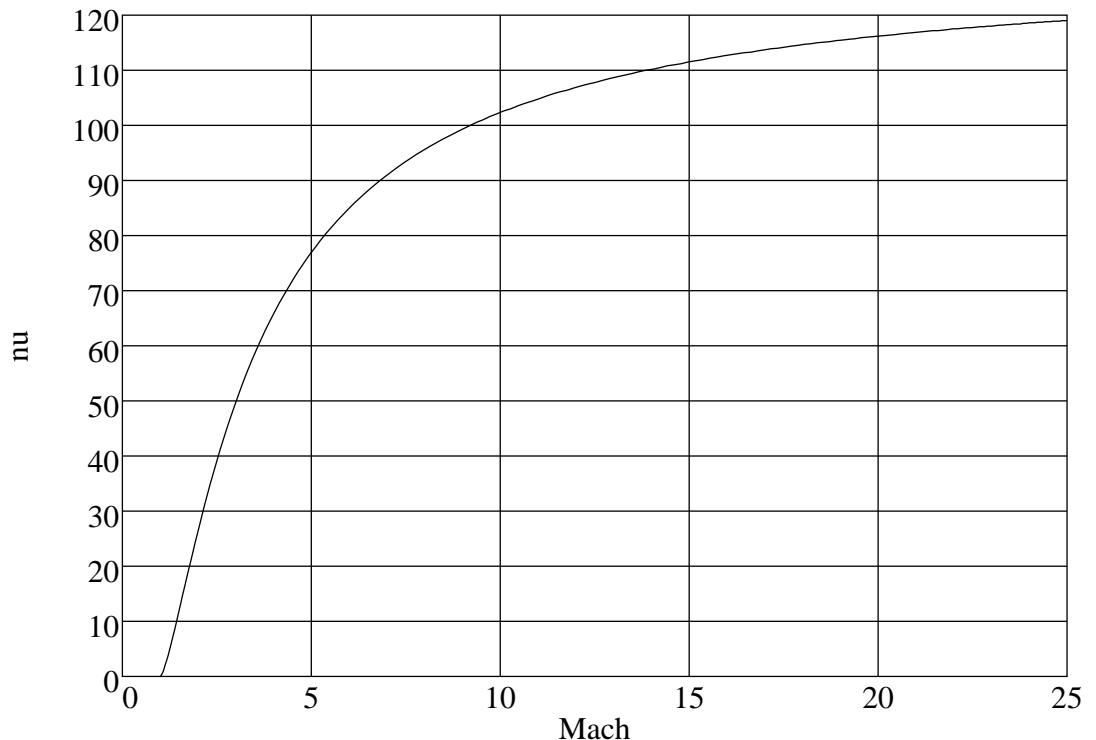


## How do you compute the inverse of the Prandtl-Meyer Function?

When a supersonic stream encounters a convex turn in which the wall is deflected away from the stream, an isentropic expansion occurs by way of a conical set of rays centered on the corner. The Prandtl-Meyer equation expresses the relation between the turning angle and the upstream and downstream Mach numbers.

$$\nu = \frac{1}{\lambda} \tan^{-1}(\lambda\beta) - \tan^{-1}(\beta)$$

where  $\lambda = \sqrt{\frac{\gamma-1}{\gamma+1}}$  and  $\beta = \sqrt{M^2-1}$ .



In the usual approach for calculation, you compute the value of  $\nu$  for the upstream Mach number and then add the angle thru which the flow is turned. The downstream Mach number is that which corresponds to the downstream  $\nu$ . But, there is no easy way to compute this. Since the equation is relatively simple, most students of compressible flow theory try to use algebraic and trigonometric manipulations to obtain an expression for Mach as a function of  $\nu$ . Alas, it does not seem possible and we are left with numerical procedures.

Given a sufficiently dense table of  $\nu$  vs. Mach, one can do a reverse table lookup. This is probably the method of choice for students who are doing an off-line calculation in simple shock-expansion theory. If one is developing a computing procedure this approach can still be used by building a large table of  $\nu$  versus Mach and then doing interpolation as Mach versus  $\nu$ .

There is a certain lack of elegance to this approach and people would like a nice neat function that simply returned Mach for a given  $\nu$ . In September 1975, I.M. Hall published a paper in the Aeronautical Journal in which he described an rational polynomial function that approximated the inverse Prandtl-Meyer function with accuracy sufficient for most engineering purposes. The function is simply a cubic polynomial divided by a quadratic.

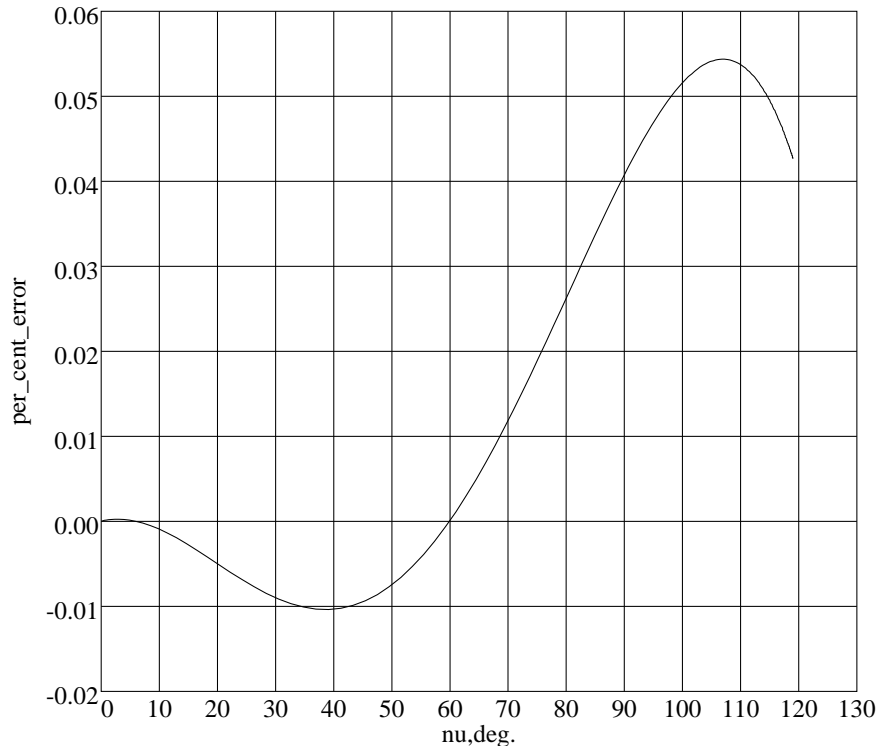
$$M = \frac{1 + Ay + By^2 + Cy^3}{1 + Dy + Ey^2}$$

where  $y = \left(\frac{\nu}{\nu_\infty}\right)^{2/3}$  and  $\nu_\infty = \frac{\pi}{2}(\sqrt{6} - 1)$ , the maximum turning angle.

The constants are

A	1.3604
B	0.0962
C	-0.5127
D	-0.6722
E	-0.3278

All of the constants shown here are for  $\gamma=7/5$ . The original paper by Hall describes the calculation for other specific heats. The accuracy of the Hall approximation is quite good with the maximum error much less that one-tenth of one percent.



This relationship of Hall provides an excellent approximation to the inverse of the Prandtl-Meyer function, but suppose you have an application that requires a very

high level of accuracy. You can use the Hall approximation as a starting point and refine the solution with Newton's method. To do this, you will need the derivative of  $\nu$  with respect to Mach.

$$\frac{d\nu}{d\beta} = \frac{\beta^2(1 - \lambda^2)}{(1 + \beta^2)(1 + \lambda^2\beta^2)}$$

and

$$\frac{d\beta}{dM} = \frac{M}{\beta}$$

which give

$$\frac{d\nu}{dM} = \frac{\beta(1 - \lambda^2)}{M(1 + \lambda^2\beta^2)}$$

I have built a module with all of the procedures described, namely the direct Prandtl-Meyer function, the inverse using Hall's approximation, and the high accuracy procedure using Newton. You can download the module of Prandtl-Meyer methods along with a test program from <http://www.pdas.com/programs/pm.f90> as a plain text file.

The source file is `pm.tex`

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