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A METHOD FOR EXPANDING A DIRECTION COSINE MATRIX INTO AN EULER SEQUENCE OF ROTATIONS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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A METHOD FOR EXPANDING A DIRECTION COSINE MATRIX

INTO AN EULER SEQUENCE OF ROTATIONS

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SUMMARY

A method has been developed for converting a matrix of direction cosines into an equivalent Euler sequence. For a desired rotational sequence, the analysis produced a set of five equations which require a sequential calculation of the Euler angles. The first angle is used to compute the second and the second is used to compute the third. For a given direction cosine matrix, the equations show that there are two Euler angle sets which will generate that matrix and can be considered to be the result of either a "positive" or a "negative" initial rotation. The results of the analysis will work throughout the entire 360° angle range and also for the singular cases. For singular cases the equations become indeterminate but the problem can be resolved if one recognizes that the two angles not involved in the singular condition simply add directly and that only their sum affects the direction cosines.

The results of the analysis have been generalized into two sets of equations, one of which applies to the classical or repeating sequences and the other to the nonclassical or nonrepeating sequences. These equations have been written as a Fortran IV subroutine and are presented in the appendixes of the report.

INTRODUCTION

In a research program for a satellite attitude control system it was found advantageous to write the equations of motion with the direction cosine matrix as the kinematic variable. If parameters such as Euler angles are used for these variables, as is often done, it is necessary to be able to convert the direction cosine output to Euler angles and the reverse. The conversion of Euler angles to direction cosines is simple and can be performed by the multiplication of elementary rotation matrices or by the use of a standard (e.g., refs. 1, 2, and 3) which gives the direction cosine matrix in terms of Euler angles. It is clear that the calculation of the Euler angles from the direction cosines is also possible, but a general method was by no means obvious from inspection of the equations involved and no reference could be found that gave a general method. In this report, a technique is given for performing the conversion. The results are given such that if a direction cosine matrix is specified and a rotation sequence given, the Euler angle sequence that will produce the direction cosines can be computed. The results for the technique have been written as a Fortran IV program presented in the appendix.

TABLE OF SYMBOLS

Analysis Section

a_{ij}	element of the direction cosine matrix
A	direction cosine matrix, 3×3
$E_3(\theta_1)$	typical elementary rotation matrix; first rotation angle θ_1, about axis 3
δ_2	eigenvector of the rotation subscripted
θ	rotation angle

Subscripts and Superscripts

1,2,3	coordinate axis for the rotation when used as a subscript for E; order of rotation of the angles when used as a subscript for θ
t	transpose of matrix
I,J,K	rotation sequence

Computer Program

TH(I)	angle θ_i
I,J,K	rotation sequence
A(I,J)	a_{ij}
EULANG	conversion subroutine name
ARTN(X(I),Y(I))	arctangent function routine, $\tan^{-1}\left(\frac{X(I)}{Y(I)}\right)$

ANALYSIS

The equations for converting the direction cosines to Euler angles will be derived for a specific Euler sequence and will then be generalized for any sequence. For a 1,2,3 rotation sequence, the direction cosines matrix can be generated by the multiplication of three elementary rotation matrices.

$$A = E_3(\theta_3)E_2(\theta_2)E_1(\theta_1) \tag{1}$$

The elementary rotations are given by

$$\left.\begin{aligned}
E_1(\theta_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \\
E_2(\theta_2) &= \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \\
E_3(\theta_3) &= \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}\right\} \tag{2}$$

Equation (1) may be written in the following form

$$E_3^t(\theta_3)\ A\ E_1^t(\theta_1) = E_2(\theta_2) \tag{3}$$

The coordinates of the eigenvector of the middle rotation are given by

$$\delta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{4}$$

If both sides of equation (3) are multiplied by δ_2, the following equation results

$$E_3^t(\theta_3)\ A\ E_1^t(\theta_1)\delta_2 = E_2(\theta_2)\delta_2 = \delta_2 \tag{5}$$

Equation (5) may be rewritten in the form

$$A\ E_1^t(\theta_1)\delta_2 = E_3(\theta_3)\delta_2 \tag{6}$$

If the following two definitions are made for the portions of equation (6),

$$X(\theta_1) = E_1^t(\theta_1)\delta_2 = \begin{pmatrix} 0 \\ \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} \tag{7}$$

$$Y(\theta_3) = E_3(\theta_3)\delta_2 = \begin{pmatrix} \sin\theta_3 \\ \cos\theta_3 \\ 0 \end{pmatrix} \tag{8}$$

Then, equation (6) may be written in the form below, where θ_2 has been eliminated from consideration.

$$A\ X(\theta_1) = Y(\theta_3) \tag{9}$$

Equation (9) is a set of three equations

$$a_{32}\cos\theta_1 + a_{33}\sin\theta_1 = 0 \tag{10}$$

$$\left.\begin{aligned} a_{12}\cos\theta_1 + a_{13}\sin\theta_1 &= \sin\theta_3 \\ a_{22}\cos\theta_1 + a_{23}\sin\theta_1 &= \cos\theta_3 \end{aligned}\right\} \tag{11}$$

And from equation (3), by expanding the left-hand side and equating the a_{11} and a_{13} terms to the corresponding terms on the right-hand side, we obtain

$$\left.\begin{aligned} a_{11}\cos\theta_3 - a_{21}\sin\theta_3 &= \cos\theta_2 \\ a_{31} &= \sin\theta_2 \end{aligned}\right\} \tag{12}$$

The desired Euler angles can be calculated from equations (10) through (12). It is necessary first to solve for θ_1 from equation (10), then for θ_3 from equation (11), and finally for θ_2 from equation (12). Note that all nine elements are used.

The solution of equations (10) through (12) results in two sets of Euler angles because the solution of equation (10) is double valued. For each value of θ_1, unique values of θ_3 and θ_2 are found from equations (11) and (12) so that two distinct Euler sequences result for a given direction cosine matrix. Physically the result means that it is possible to reach a given position by starting with either a "positive" or a "negative" rotation θ_1 followed by the appropriate rotations, θ_2 and θ_3, to give the final position. For a non-classical sequence such as the one used in the previous analysis (1,2,3), the relation between the two sequences is as shown below.

$$\left.\begin{array}{llll} \text{case 1} & \theta_1 , & \theta_2 , & \theta_3 \\ \text{case 2} & \theta_1 \pm \pi , & -\theta_2 \pm \pi , & \theta_3 \pm \pi \end{array}\right\} \qquad (13)$$

For the classical or repeating type sequence, the relation between the angles is different and can be shown to be of the following form:

$$\left.\begin{array}{llll} \text{case 1} & \theta_1 , & \theta_2 , & \theta_3 \\ \text{case 2} & \theta_1 \pm \pi , & -\theta_2 , & \theta_3 \pm \pi \end{array}\right\} \qquad (14)$$

For each Euler sequence there is a singular case which requires special handling. The repeating sequence is singular when sine θ_2 is zero and the nonrepeating sequence is singular when cosine θ_2 is zero. When the 1,2,3 sequence is singular, the direction cosines a_{32} and a_{33} are both zero and the computation of θ_1 is indeterminate. The problem can be resolved by noting that θ_1 and θ_3 add directly. Since the direction cosine matrix only specifies the final position, θ_1 and θ_3 may have any value so long as their sum is correct. In the computation it is necessary to set θ_1 to some value, possibly zero, and then proceed.

Once equations (10) through (12) have been derived, it is easy to see how they could have been obtained by direct inspection of the direction cosine matrix written in terms of Euler angles. For the 1,2,3 rotation sequence the matrix is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta_2 \cos\theta_3 & \begin{matrix}\cos\theta_1 \sin\theta_3 \\ + \sin\theta_1 \sin\theta_2 \cos\theta_3\end{matrix} & \begin{matrix}\sin\theta_1 \sin\theta_3 \\ - \cos\theta_1 \sin\theta_2 \cos\theta_3\end{matrix} \\ -\cos\theta_2 \sin\theta_3 & \begin{matrix}\cos\theta_1 \cos\theta_3 \\ - \sin\theta_1 \sin\theta_2 \sin\theta_3\end{matrix} & \begin{matrix}\sin\theta_1 \cos\theta_3 \\ + \cos\theta_1 \sin\theta_2 \sin\theta_3\end{matrix} \\ \sin\theta_2 & -\sin\theta_1 \cos\theta_2 & \cos\theta_1 \cos\theta_2 \end{vmatrix} \qquad (18)$$

The following operations

$$\left.\begin{array}{lll} a_{32} = -\sin\theta_1 \cos\theta_2 \ , & \text{multiply by} & \cos\theta_1 \\ a_{33} = \ \ \cos\theta_1 \cos\theta_2 \ , & \text{multiply by} & \sin\theta_1 \end{array}\right\} \tag{19}$$

and adding the two equations yields equation (10)

$$a_{32} \cos\theta_1 + a_{33} \sin\theta_1 = 0 \tag{20}$$

Next,

$$a_{31} = \sin\theta_2$$

together with the following operations

$$\left.\begin{array}{lll} a_{12} = \ \ \cos\theta_2 \cos\theta_3 \ , & \text{multiply by} & \cos\theta_3 \\ a_{21} = -\cos\theta_2 \sin\theta_3 \ , & \text{multiply by} & -\sin\theta_3 \end{array}\right\} \tag{21}$$

and adding the last two equations, yields equations (12).

$$\left.\begin{array}{l} a_{11} \cos\theta_3 - a_{21} \sin\theta_3 = \cos\theta_2 \\ a_{31} = \sin\theta_2 \end{array}\right\} \tag{22}$$

Finally, to obtain equations (11), perform the following operations:

$$\left.\begin{array}{lll} a_{12} = \cos\theta_1 \sin\theta_3 + \sin\theta_1 \sin\theta_2 \cos\theta_3 \ , & \text{multiply by} & \cos\theta_1 \\ a_{13} = \sin\theta_1 \sin\theta_3 - \cos\theta_1 \sin\theta_2 \cos\theta_3 \ , & \text{multiply by} & \sin\theta_1 \end{array}\right\} \tag{23}$$

and the two equations add to get

$$a_{12} \cos\theta_1 + a_{13} \sin\theta_1 = \sin\theta_3 \tag{24}$$

Similarly,

$$\left.\begin{array}{lll} a_{22} = \cos\theta_1 \cos\theta_3 - \sin\theta_1 \sin\theta_2 \sin\theta_3 \ , & \text{multiply by} & \cos\theta_1 \\ a_{23} = \sin\theta_1 \cos\theta_3 + \cos\theta_1 \sin\theta_2 \sin\theta_3 \ , & \text{multiply by} & \sin\theta_1 \end{array}\right\} \tag{25}$$

yields

$$a_{22} \cos\theta_1 + a_{23} \sin\theta_1 = \cos\theta_3 \tag{26}$$

The resulting equations (24) and (26) are equations (11).

An inspection of a direction cosine matrix written in terms of any other Euler sequence will show that the same procedure can be followed with the proper choice of terms. Also it should be noted that the solution sequence could be in reverse order, θ_3, θ_1, θ_2 instead of θ_1, θ_3, θ_2.

The derivation that resulted in equations (10) through (12) can be generalized to apply to any Euler sequence since $E_J(\theta_2)\delta_J = \delta_J$. If the rotation sequence is I, J, K, the direction cosine matrix will be given by

$$A = E_K(\theta_3)E_J(\theta_2)E_I(\theta_1) \tag{27}$$

The first step is to form the eigenvector of the middle rotation, δ_J, and then compute the column vectors

$$\left.\begin{aligned} X(\theta_1) &= E_I^{\,t}(\theta_1)\delta_J \\ Y(\theta_3) &= E_K(\theta_3)\delta_J \end{aligned}\right\} \tag{28}$$

The equation to be used for the solution of θ_1 and θ_3 is then

$$A\ X(\theta_1) = Y(\theta_3) \tag{29}$$

and θ_2 can be computed by use of equation (3) rewritten in the form

$$E_J(\theta_2) = E_K^{\,t}(\theta_3)\ A\ E_I^{\,t}(\theta_1) \tag{30}$$

A set of five equations similar to equations (10) through (12) will result from the expansion of equations (29) and (30). The 12 possible sets of equations are given in appendix A.

The 12 sets of equations that result from equations (29) and (30) can be reduced to two sets of equations with an appropriate method of indexing the direction cosine elements. One of the two sets applies to the classical or repeating sequences and the other to the nonclassical or nonrepeating sequences. These equations have been written as a Fortran IV subroutine presented in appendix B. For computing, the necessary input to the subroutine is the array of nine direction cosines and the desired rotational sequence. The output will be the three Euler angles. Since there are two Euler sequences for a given direction cosine matrix, the program has been set up to compute only one. If the other is desired, equations (13) or (14) can be used for the conversion. Also, in the singular case computations, θ_1 has been set to zero. For a sequence of computations this may cause a discontinuity in the output. Smoothing may be accomplished by extrapolation of previous values of θ_1 and using this as the output value of θ_1 and subtracting the same amount from θ_3.

CONCLUDING REMARKS

In the preceding analysis a method was developed for converting a matrix of direction cosines into an Euler angle sequence. For a given rotational sequence a set of five equations can be written which call for a sequential calculation of the three Euler angles. The first angle is used in the calculation of the second and the second in the calculation of the third. The equations show that for a given direction cosine matrix and a specified rotational sequence there are two Euler angle sequences that satisfy the equations. The reason is that the particular values of the direction cosines may be produced by an initially positive rotation followed by two others of correct magnitude or an initially negative rotation and then two additional rotations. For the singular case the conversion equations become indeterminate but the first and last rotations add directly. The indeterminate conditions arise because any two values of these angles will satisfy the direction cosine matrix as long as their sum is correct. It is necessary to assume some value for the first angle (zero or some extrapolated value from previous calculations) and proceed with the computations.

The equations for the conversion have been generalized so that for a given direction cosine matrix and any desired rotational sequence the Euler angles can be computed. This set of equations has been written as a Fortran Fortran IV computer subroutine and is presented in appendix B.

Ames Research Center
National Aeronautics and Space Administration
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APPENDIX A

EQUATIONS REQUIRED TO CONVERT DIRECTION COSINES TO EULER ANGLES

FOR ALL 12 EULER SEQUENCES

1,2,3 Sequence

$$a_{33} \sin \theta_1 + a_{32} \cos \theta_1 = 0$$

$$a_{13} \sin \theta_1 + a_{12} \cos \theta_1 = \sin \theta_3$$

$$a_{23} \sin \theta_1 + a_{22} \cos \theta_1 = \cos \theta_3$$

$$a_{31} = \sin \theta_2$$

$$-a_{21} \sin \theta_3 + a_{11} \cos \theta_3 = \cos \theta_2$$

1,3,2 Sequence

$$a_{22} \sin \theta_1 - a_{23} \cos \theta_1 = 0$$

$$a_{12} \sin \theta_1 - a_{13} \cos \theta_1 = \sin \theta_3$$

$$-a_{32} \sin \theta_1 + a_{33} \cos \theta_1 = \cos \theta_3$$

$$-a_{21} = \sin \theta_2$$

$$a_{31} \sin \theta_3 + a_{11} \cos \theta_3 = \cos \theta_2$$

2,1,3 Sequence

$$-a_{33} \sin \theta_1 + a_{31} \cos \theta_1 = 0$$

$$a_{23} \sin \theta_1 - a_{21} \cos \theta_1 = \sin \theta_3$$

$$-a_{13} \sin \theta_1 + a_{11} \cos \theta_1 = \cos \theta_3$$

$$-a_{32} = \sin \theta_2$$

$$a_{12} \sin \theta_3 + a_{22} \cos \theta_3 = \cos \theta_2$$

2,3,1 Sequence

$$a_{11} \sin \theta_1 + a_{13} \cos \theta_1 = 0$$

$$a_{21} \sin \theta_1 + a_{23} \cos \theta_1 = \sin \theta_3$$

$$a_{31} \sin \theta_1 + a_{33} \cos \theta_1 = \cos \theta_3$$

$$a_{12} = \sin \theta_2$$

$$-a_{32} \sin \theta_3 + a_{22} \cos \theta_3 = \cos \theta_2$$

3,1,2 Sequence

$$a_{22} \sin \theta_1 + a_{21} \cos \theta_1 = 0$$

$$a_{32} \sin \theta_1 + a_{31} \cos \theta_1 = \sin \theta_3$$

$$a_{12} \sin \theta_1 + a_{11} \cos \theta_1 = \cos \theta_3$$

$$a_{23} = \sin \theta_2$$

$$-a_{13} \sin \theta_3 + a_{33} \cos \theta_3 = \cos \theta_2$$

3,2,1 Sequence

$$a_{11} \sin \theta_1 - a_{12} \cos \theta_1 = 0$$

$$a_{31} \sin \theta_1 - a_{32} \cos \theta_1 = \sin \theta_3$$

$$-a_{21} \sin \theta_1 + a_{22} \cos \theta_1 = \cos \theta_3$$

$$-a_{13} = \sin \theta_2$$

$$a_{23} \sin \theta_3 + a_{33} \cos \theta_3 = \cos \theta_2$$

1,2,1 Sequence

$$a_{13} \sin \theta_1 + a_{12} \cos \theta_1 = 0$$

$$a_{33} \sin \theta_1 + a_{32} \cos \theta_1 = -\sin \theta_3$$

$$a_{23} \sin \theta_1 + a_{22} \cos \theta_1 = \cos \theta_3$$

$$a_{21} \sin \theta_3 + a_{31} \cos \theta_3 = \sin \theta_2$$

$$a_{11} = \cos \theta_2$$

1,3,1 Sequence

$$a_{12} \sin \theta_1 - a_{13} \cos \theta_1 = 0$$

$$-a_{22} \sin \theta_1 + a_{23} \cos \theta_1 = \sin \theta_3$$

$$-a_{32} \sin \theta_1 + a_{33} \cos \theta_1 = \cos \theta_3$$

$$a_{31} \sin \theta_3 - a_{21} \cos {}_{31} = \sin \theta_2$$

$$a_{11} = \cos \theta_2$$

2,1,2 Sequence

$$a_{23} \sin \theta_1 - a_{21} \cos \theta_1 = 0$$

$$-a_{33} \sin \theta_1 + a_{31} \cos \theta_1 = \sin \theta_3$$

$$-a_{13} \sin \theta_1 + a_{11} \cos \theta_1 = \cos \theta_3$$

$$a_{12} \sin \theta_3 - a_{32} \cos \theta_3 = \sin \theta_2$$

$$a_{22} = \cos \theta_2$$

2,3,2 Sequence

$$a_{21} \sin \theta_1 + a_{23} \cos \theta_1 = 0$$

$$a_{11} \sin \theta_1 + a_{13} \cos \theta_1 = -\sin \theta_3$$

$$a_{31} \sin \theta_1 + a_{33} \cos \theta_1 = \cos \theta_3$$

$$a_{32} \sin \theta_3 + a_{12} \cos \theta_3 = \sin \theta_2$$

$$a_{22} = \cos \theta_2$$

3,1,3 Sequence

$$a_{32} \sin \theta_1 + a_{31} \cos \theta_1 = 0$$

$$a_{22} \sin \theta_1 + a_{21} \cos \theta_1 = -\sin \theta_3$$

$$a_{12} \sin \theta_1 + a_{11} \cos \theta_1 = \cos \theta_3$$

$$a_{13} \sin \theta_3 + a_{23} \cos \theta_3 = \sin \theta_2$$

$$a_{33} = \cos \theta_2$$

3,2,3 Sequence

$$a_{31} \sin \theta_1 - a_{32} \cos \theta_1 = 0$$

$$-a_{11} \sin \theta_1 + a_{12} \cos \theta_1 = \sin \theta_3$$

$$-a_{21} \sin \theta_1 + a_{22} \cos \theta_1 = \cos \theta_3$$

$$a_{23} \sin \theta_3 - a_{13} \cos \theta_3 = \sin \theta_2$$

$$a_{33} = \cos \theta_2$$

APPENDIX B

FORTRAN IV PROGRAM FOR DETERMINING EULER ANGLES FROM A GIVEN DIRECTION COSINE MATRIX

```
      SUBROUTINE EULANG (I,J,K,A,TH)
      DIMENSION X(3), Y(3), TH(3), A(3,3)
      IF (I.EQ.K) GO TO 103
      L = I - MOD(J,3)
      IF (L.EQ.2) L = -1
      C = L
      X(1) = A(K,J)*C
      Y(1) = A(K,K)
      TH(1) = ARTN(X(1),Y(1))
      X(1) = SIN(TH(1))
      Y(1) = COS(TH(1))
  101 X(3) = A(I,K)*X(1) - A(I,J)*C*Y(1)
      Y(3) = A(J,J)*Y(1) - A(J,K)*C*X(1)
      X(2) = -A(K,I)*C
      Y(2) = A(I,I)*Y(3) + A(J,I)*C*X(3)
      GO TO 104
  103 N = 6 - (K + J)
      X(1) = A(K,J)
      L = N - MOD(I,3)
      IF (L.EQ.2) L = -1
      C = L
      Y(1) = A(K,N)*C
      TH(1) = ARTN(X(1),Y(1))
      X(1) = SIN(TH(1))
      Y(1) = COS(TH(1))
  102 X(3) = -A(N,N)*X(1) + A(N,J)*C*Y(1)
      Y(3) = A(J,J)*Y(1) - A(J,N)*C*X(1)
      X(2) = A(J,I)*X(3) - A(N,I)*C*Y(3)
      Y(2) = A(K,K)
  104 TH(3) = ARTN(X(3),Y(3))
      TH(2) = ARTN(X(2),Y(2))
      RETURN
      END
```

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