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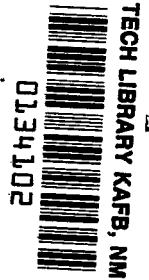
AN ALGORITHM FOR THE WEIGHTING MATRICES IN THE SAMPLED-DATA OPTIMAL LINEAR REGULATOR PROBLEM

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16. Abstract

The sampled-data optimal linear regulator problem provides a means whereby a control designer can use an understanding of continuous optimal regulator design to produce a digital state variable feedback control law which satisfies continuous system performance specifications. A basic difficulty in applying the sampled-data regulator theory is the requirement that certain digital performance index weighting matrices, expressed as complicated functions of system matrices, be computed. This report presents infinite series representations for the weighting matrices of the time-invariant version of the optimal linear sampled-data regulator problem. Error bounds are given for estimating the effect of truncating the series expressions after a finite number of terms and a method is described for their computer implementation. A numerical example is given to illustrate the results.

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SUMMARY

The sampled-data optimal linear regulator problem provides a means whereby a control designer can use an understanding of continuous optimal regulator design to produce a digital state variable feedback control law which satisfies continuous system performance specifications. A basic difficulty in applying the sampled-data regulator theory is the requirement that certain digital performance index weighting matrices, expressed as complicated functions of system matrices, be computed. This report presents infinite series representations for the weighting matrices of the time-invariant version of the optimal linear sampled-data regulator problem. Error bounds are given for estimating the effect of truncating the series expressions after a finite number of terms and a method is described for their computer implementation. A numerical example is given to illustrate the results.

INTRODUCTION

Optimal linear quadratic regulator theory, currently referred to as the Linear Quadratic Gaussian (LQG) problem (ref. 1), has become one of the most widely accepted methods for determining optimal control policy. In the continuous dynamics version of the LQG problem, the system to be controlled is modeled as a system of first-order vector-matrix ordinary differential equations linear in the state and control variables. An optimal linear state variable feedback control law is obtained from the minimization of an integral performance index whose integrand is composed of weighted quadratic terms in the state and control. The user adjusts the weights to cause the closed loop response of the dynamic system to satisfy required design specifications. (See refs. 2 and 3.) A discrete dynamics analog of the continuous LQG problem also exists (ref. 4) in which the state equations are first-order vector-matrix finite-difference equations linear in the state and control variables evaluated at distinct time points. Both the continuous and discrete versions of the LQG problem provide the control designer with a rigorous tool for developing linear state variable feedback control laws for multi-input multi-output dynamical systems.

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In many control applications the physical system to be controlled is modeled by using continuous dynamic equations, and the control design specifications are given in terms of response criteria for continuous systems, but the control law is to be implemented in a digital fashion. A typical example is aircraft flight control applications where an onboard digital computer is used to generate the controller commands. In such cases a problem arises as to how to find a digital control law which achieves continuous design specifications. One approach, which is in the context of the LQG methodology, is the Optimal Sampled-Data Regulator (OSR) problem (refs. 4 and 5). The OSR problem is a continuous dynamics LQG problem with one additional constraint. The operational time interval is divided into segments (sampling intervals) within which the control function takes on constant values. In other words, the OSR problem is a continuous LQG problem where the control is required to be piecewise constant. The OSR problem can easily be transformed into a discrete LQG problem (ref. 6) and solved by using standard LQG solution algorithms. The weighting matrices in the continuous OSR performance index are chosen to achieve the continuous design specifications as if a continuous LQG solution is to be implemented and then transformed into equivalent weighting matrices for use in the discrete LQG problem computations. The transformation equations produce extra control-state cross-product terms in the digital performance index and off-diagonal entries in the weighting matrices for the other terms which have the effect of weighting the state and control variables at points within the sampling interval as well as at the sampling instants. The OSR approach provides a means whereby the users' intuitive understanding of continuous design problems and LQG techniques can be applied to produce digital feedback control laws which satisfy continuous performance specifications.

A primary difficulty in implementing the OSR problem is that the transformation equations defining the digital weighting matrices are complicated functions of the continuous system state transition matrix and other matrices appearing in the continuous formulation. In fact, the computation of these matrices for high order systems often discourages the use of the OSR methodology. In this report this difficulty is eliminated for the time-invariant version of the OSR problem by presenting a general purpose, numerically attractive, easy-to-implement algorithm for computing the discrete weighting matrices. For the time-invariant OSR problem, the transformed weighting matrices are noted to be analytic functions of the sampling interval. Applying the analyticity property yields infinite series expressions for the transformation equations. Error bounds are presented that show the effect of truncating the series solutions after a finite number of terms and a technique is recommended for implementing the algorithm on a digital computer. Finally, the results of this paper are demonstrated with a numerical example.

2

A   open-loop response matrix for continuous linear system

B   control effectiveness matrix for continuous linear system

$C_Q(j)$ matrix defined by equation (21)

$C_R(j)$ matrix defined by equation (59)

$C_W(j)$ matrix defined by equation (38)

$D(j)$ matrix defined by equation (43)

$E(\ell)$ $= \dfrac{2^{(\ell+1)} \|A\|^{(\ell+1)} \left(\Delta t_i\right)^{(\ell+2)} \|Q_c\| e^{2\|A\|\Delta t_i}}{(\ell + 2)!}$ where ℓ is a natural number

$E_Q(n_1), E_W(n_2), E_{\tilde{R}}(n_3)$ error after truncating Q_d, W, and $\tilde{R}_d$ series after n_1, n_2, and n_3 terms, respectively

e^c exponential of argument c

F   matrix defined by equation (B2)

$f_1, \ldots f_6$ functions defined by equations (B3) to (B8)

$G(j)$ matrix defined by equation (62)

$H(a,b)$ $= \displaystyle\int_0^{a-b} e^{A\tau}\,d\tau\, B$ where a and b are parameters

I $n \times n$ identity matrix

i,j   indices

K   positive integer satisfying condition (71)

$L(\tau)$ matrix defined by equation (58)

N	number of sampling points
n	order of matrix A
n_1, n_2, n_3	number of terms at which summation index for Q_d, W, and $\tilde{R}_d$ series are truncated, respectively
Q_c	constant nonnegative definite symmetric matrix
$\hat{Q}_d(i)$	matrix defined by equation (6) or (11)
$Q_d\left(\Delta t_i\right)$	matrix defined by equation (11)
R_c	constant symmetric positive definite matrix
$\hat{R}_d(i)$	matrix defined by equation (8) or (13)
$R_d\left(\Delta t_i\right)$	matrix defined by equation (13)
$\tilde{R}_d\left(\Delta t_i\right)$	matrix defined by equation (54)
r	number of columns in matrix B
S	constant symmetric nonnegative definite matrix
T	final value of time
t	time variable, $0 \leq t \leq T$
t_i	ordered set of points within $[0,T]$ so that $t_o < t_i < \ldots < t_N$
t_N	T, final time
t_o	0, initial time
u(t)	control vector
$V(\tau)$	matrix defined by equation (34)
$\hat{W}(i)$	matrix defined by equation (7) or (12)

4

$W\left(\Delta t_i\right)$ matrix defined by equation (12)

$X(\tau)$ matrix defined by equation (17)

$x(t)$ state vector

$Y(j)$ matrix defined by equation (25)

z, τ, w integration variables

Δ $= \Delta t_i \big/ 2^K$

Δt_i sample interval, $= t_{i+1} - t_i$

$\| \; \|$ a matrix operator norm such that $\|C\| = \|C'\|$ for a matrix C

Subscripts:

c   continuous dynamics

d   discrete dynamics

i   ith stage

N   final stage

o   initial stage

Q   related to. Q_d computation

R   related to R_d computation

$\tilde{R}$ related to $\tilde{R}_d$ computation

W   related to   W   computation

Primes denote a matrix transpose. A dot denotes differentiation with respect to   t.

5

THE TIME-INVARIANT SAMPLED-DATA OPTIMAL
LINEAR REGULATOR PROBLEM

If the linear time-invariant system

$$\dot{x} = Ax + Bu \qquad\qquad \left(x(0) = x_o\right) \qquad (1)$$

is given where the matrices A and B are constant and of order $n \times n$ and $n \times r$, respectively, the sampled-data regulator problem occurs when the control function $u(t)$ $(0 \leqq t \leqq T)$ is required to minimize the functional J

$$J = x'(T)\, S\, x(T) + \int_0^T \left[x'(\tau)\, Q_c\, x(\tau) + u'(\tau)\, R_c\, u(\tau) \right] d\tau \qquad (2)$$

subject to the restriction that $u(t)$ be constant over subintervals $t_0 < t_1 < \ldots < t_N$ of $[0, T]$ where

$$t_o = 0$$

and

$$t_N = T$$

That is,

$$u(t) = u\left(t_i\right) \qquad \left(t_i \leqq t < t_{i+1} \quad \text{where} \quad i = 0, 1, \ldots, N - 1\right) \qquad (3)$$

The matrices S, Q_c, and R_c of equation (2) satisfy

$$S = S' \geqq 0$$

$$Q_c = Q_c' \geqq 0$$

and

$$R_c = R_c' > 0$$

Applying condition (3) to equation (1) gives

$$x(t) = e^{A(t-t_i)} x(t_i) + \int_{t_i}^{t} e^{A(t-\tau)} \, d\tau \, B \, u(t_i) \qquad \left(t_i \leqq t < t_{i+1} \right) \qquad (4)$$

Substituting equation (4) into equation (2) yields

$$J = x'(t_N) \, S \, x(t_N) + \sum_{i=0}^{N-1} \left[x'(t_i) \, \hat{Q}_d(i) \, x(t_i) + x'(t_i) \, \hat{W}(i) \, u(t_i) \right.$$

$$\left. + u'(t_i) \, \hat{R}_d(i) \, u(t_i) \right] \qquad (5)$$

with

$$\hat{Q}_d(i) = \int_{t_i}^{t_{i+1}} e^{A'(\tau - t_i)} \, Q_c \, e^{A(\tau - t_i)} \, d\tau \qquad (6)$$

$$\frac{\hat{W}(i)}{2} = \int_{t_i}^{t_{i+1}} e^{A'(\tau - t_i)} \, Q_c \, H(\tau, t_i) \, d\tau \qquad (7)$$

$$\hat{R}_d(i) = \int_{t_i}^{t_{i+1}} \left[R_c + H'(\tau, t_i) \, Q_c \, H(\tau, t_i) \right] d\tau \qquad (8)$$

and

$$H(t, t_i) = \int_{t_i}^{t} e^{A(t-\tau)} \, d\tau \, B \qquad (9)$$

In terms of

$$\Delta t_i = t_{i+1} - t_i \qquad (10)$$

equations (6) to (9) become

7

$$\hat{Q}_d(i) = Q_d\left(\Delta t_i\right) = \int_0^{\Delta t_i} e^{A'\tau} Q_c\, e^{A\tau}\, d\tau \tag{11}$$

$$\frac{\hat{W}(i)}{2} = \frac{W\left(\Delta t_i\right)}{2} = \int_0^{\Delta t_i} e^{A'\tau} Q_c\, H(\tau,0)\, d\tau \tag{12}$$

$$\hat{R}_d(i) = R_d\left(\Delta t_i\right) = \int_0^{\Delta t_i} \left[R_c + H'(\tau,0)\, Q_c\, H(\tau,0)\right] d\tau \tag{13}$$

$$H(t,0) = \int_0^t e^{A\tau}\, d\tau\; B \tag{14}$$

The sampled-data regulator problem then becomes the standard discrete linear quadratic regulator problem of choosing $u\left(t_i\right)$ (i = 0, 1, . . ., N - 1) to minimize equation (5) subject to equations (11) to (14) and

$$x\left(t_{i+1}\right) = e^{A\,\Delta t_i}\, x\left(t_i\right) + H\left(\Delta t_i,0\right) u\left(t_i\right) \qquad \left(x\left(t_o\right) = x_o\right) \tag{15}$$

the solution of which can be obtained through well-known methods (ref. 4) once the matrices $Q_d\left(\Delta t_i\right)$, $W\left(\Delta t_i\right)$, and $R_d\left(\Delta t_i\right)$ are computed. The following sections of this paper present an algorithm for efficiently computing $Q_d\left(\Delta t_i\right)$, $W\left(\Delta t_i\right)$, and $R_d\left(\Delta t_i\right)$ for arbitrary (constant) A, B, Q_c, R_c, and Δt_i.

EVALUATION OF $Q_d\left(\Delta t_i\right)$

Recall from equation (11) that

$$Q_d\left(\Delta t_i\right) = \int_0^{\Delta t_i} X(\tau)\, d\tau \tag{16}$$

where

$$X(\tau) = e^{A'\tau} Q_c\, e^{A\tau} \tag{17}$$

is analytic in τ and hence can be expanded in a convergent Taylor series about $\tau = 0$

$$X(\tau) = \sum_{j=0}^{\infty} \left. \frac{d^j X(\tau)}{d\tau^j} \right|_{\tau=0} \frac{\tau^j}{j!} \qquad (18)$$

Noting that

$$\frac{dX(\tau)}{d\tau} = A'X(\tau) + X(\tau) A \qquad (19)$$

gives

$$\frac{d^{j+1}X(\tau)}{d\tau^{j+1}} = A' \frac{d^j X(\tau)}{d\tau^j} + \frac{d^j X(\tau)}{d\tau^j} A \qquad (j = 0, 1, \ldots) \qquad (20)$$

and equation (18) can be rewritten as

$$X(\tau) = \sum_{j=0}^{\infty} C_Q(j) \frac{\tau^j}{j!} \qquad (21)$$

where

$$C_Q(j + 1) = A' C_Q(j) + C_Q(j) A \qquad \left(C_Q(0) = Q_c\right) \qquad (22)$$

From equations (16) and (21),

$$Q_d\left(\Delta t_i\right) = \int_0^{\Delta t_i} X(\tau) \, d\tau \qquad (23)$$

$$Q_d\left(\Delta t_i\right) = \sum_{j=0}^{\infty} C_Q(j) \frac{\left(\Delta t_i\right)^{j+1}}{(j+1)!} \qquad (24)$$

For computational purposes, let

$$Y(j) = C_Q(j) \frac{\left(\Delta t_i\right)^{j+1}}{(j+1)!} \tag{25}$$

From equation (22)

$$Y(j+1) = \left(\frac{A' \Delta t_i}{j+2}\right) Y(j) + Y(j)\left(\frac{A \Delta t_i}{j+2}\right) \qquad \left(j = 0, 1, \ldots; \quad Y(0) = Q_c \Delta t_i\right) \tag{26}$$

whereby

$$Q_d\left(\Delta t_i\right) = Y(0) + Y(1) + \ldots \tag{27}$$

In practice, equation (24) or (27) is truncated after a finite number of terms, for example, n_1. From equation (22),

$$\left\|C_Q(j+1)\right\| \leqq 2\left\|A\right\| \left\|C_Q(j)\right\| \qquad \left(\left\|C_Q(0)\right\| = \left\|Q_c\right\|\right) \tag{28}$$

whereby

$$\left\|C_Q(j)\right\| \leqq 2^j \left\|A\right\|^j \left\|Q_c\right\| \qquad (j = 0, 1, \ldots) \tag{29}$$

Also, with equation (28)

$$\left\|\sum_{j=n_1+1}^{\infty} C_Q(j) \frac{\left(\Delta t_i\right)^{j+1}}{(j+1)!}\right\| \leqq \left\|Q_c\right\| \sum_{j=n_1+1}^{\infty} 2^j \left\|A\right\|^j \frac{\left(\Delta t_i\right)^{j+1}}{(j+1)!}$$

$$= 2^{\left(n_1+1\right)} \left\|A\right\|^{\left(n_1+1\right)} \left(\Delta t_i\right)^{\left(n_1+2\right)} \left\|Q_c\right\| \sum_{j=0}^{\infty} \frac{2^j \left\|A\right\|^j \left(\Delta t_i\right)^j}{\left(j+n_1+2\right)!}$$

$$= \frac{2^{(n_1+1)} \|A\|^{(n_1+1)} (\Delta t_i)^{(n_1+2)} \|Q_c\|}{(n_1+2)!} \sum_{j=0}^{\infty} 2^j \|A\|^j (\Delta t_i)^j \frac{(n_1+2)!}{(j+n_1+2)!}$$

$$< \frac{2^{(n_1+1)} \|A\|^{(n_1+1)} (\Delta t_i)^{(n_1+2)} \|Q_c\|}{(n_1+2)!} e^{2\|A\|\Delta t_i} \tag{30}$$

Therefore, by denoting the terms in $Q_d(\Delta t_i)$ after term n_1 by $E(n_1)$,

$$Q_d(\Delta t_i) = \sum_{j=0}^{n_1} \frac{C_Q(j) (\Delta t_i)^{j+1}}{(j+1)!} + E_Q(n_1) \tag{31}$$

with

$$\|E_Q(n_1)\| < E(n_1) = \frac{2^{(n_1+1)} \|A\|^{(n_1+1)} (\Delta t_i)^{(n_1+2)} \|Q_c\|}{(n_1+2)!} e^{2\|A\|\Delta t_i} \tag{32}$$

The function $e(n_1)$ provides an upper bound on the error obtained by truncating the $Q_d(\Delta t_i)$ series after n_1 terms.

EVALUATION OF $W(\Delta t_i)$

Since, from equation (12),

$$\frac{W(\Delta t_i)}{2} = \int_0^{\Delta t_i} V(\tau) \, d\tau \tag{33}$$

with

$$V(\tau) = e^{A'} Q_c \int_0^\tau e^{Az} B \, dz \tag{34}$$

analytic in τ, $V(\tau)$ can be written as

$$V(\tau) = \sum_{j=0}^{\infty} \frac{d^j V(\tau)}{d\tau^j}\bigg|_{\tau=0} \frac{\tau^j}{j!} \tag{35}$$

Differentiating $V(\tau)$ from equation (34) gives

$$\frac{dV(\tau)}{d\tau} = A' \, V(\tau) + X(\tau) \, B \tag{36}$$

where $X(\tau)$ is given by equation (17). Recursively,

$$\frac{d^{j+1}}{d\tau^{j+1}} V(\tau) = A' \frac{d^j V(\tau)}{d\tau^j} + \frac{d^j X(\tau)}{d\tau^j} B \qquad (j = 0, 1, \ldots) \tag{37}$$

Defining

$$C_W(j) = \frac{d^j V(\tau)}{d\tau^j}\bigg|_{\tau=0} \tag{38}$$

gives, from equation (34),

$$V(\tau) = \sum_{j=0}^{\infty} C_W(j) \frac{\tau^j}{j!} \tag{39}$$

where for $j = 0, 1, \ldots,$

$$C_W(j + 1) = A' \, C_W(j) + C_Q(j) \, B \qquad \left(C_W(0) = 0; \quad C_Q(0) = Q_c \right) \tag{40}$$

Equation (33) yields

$$\frac{W\left(\Delta t_i\right)}{2} = \int_0^{\Delta t_i} V(\tau) \, d\tau \tag{41}$$

12

or

$$\frac{W\left(\Delta t_i\right)}{2} = \sum_{j=1}^{\infty} C_W(j) \frac{\left(\Delta t_i\right)^{j+1}}{(j+1)!} \tag{42}$$

For computational purposes, let

$$D(j) = C_W(j) \frac{\left(\Delta t_i\right)^{j+1}}{(j+1)!} \tag{43}$$

Then for $j = 0, 1, \ldots,$

$$D(j+1) = \left(\frac{A' \Delta t_i}{j+2}\right) D(j) + Y(j)\left(\frac{B \Delta t_i}{j+2}\right) \quad \left(D(0) = 0; \quad Y(0) = Q_c \Delta t_i\right) \tag{44}$$

where $Y(j)$ is computed from equation (26). In terms of D,

$$W\left(\Delta t_i\right) = 2\left[D(1) + D(2) + \ldots\right] \tag{45}$$

The computation of an upper bound on the error in truncating equation (42) after a finite number of terms, for example, $j = n_2 \geq 1$, is somewhat more complicated than the corresponding computation for $Q_d\left(\Delta t_i\right)$ but can still be carried out by using the same general approach. From equations (29) and (40),

$$\left\|C_W(j+1)\right\| \leq \left\|A\right\| \left\|C_W(j)\right\| + \left\|C_Q(j)\right\| \left\|B\right\|$$

$$\leq \left\|A\right\| \left\|C_W(j)\right\| + 2^j \left\|A\right\|^j \left\|Q_c\right\| \left\|B\right\| \tag{46}$$

Rearranging equation (46) and assuming that $\left\|A\right\| \neq 0$ yields

$$\left\|A\right\|^{-j}\left\|C_W(j+1)\right\| - \left\|A\right\|^{-j+1}\left\|C_W(j)\right\| \leq 2^j\left\|Q_c\right\| \left\|B\right\| \tag{47}$$

Summing inequality (47) from $j = 0$ to $j - 1$ yields

$$\left\|A\right\|^{-j+1}\left\|C_W(j)\right\| - \left\|A\right\| \left\|C_W(0)\right\| \leq \left\|Q_c\right\| \left\|B\right\|\left(2^j - 1\right)$$

13

or

$$\left\|C_W(j)\right\| \leqq \left\|Q_c\right\| \, \|B\| \, \|A\|^{j-1}\left(2^j - 1\right) \tag{48}$$

Writing

$$\frac{W\left(\Delta t_i\right)}{2} = \sum_{j=0}^{n_2} \frac{C_W(j+1)\left(\Delta t_i\right)^{j+2}}{(j+2)!} + \sum_{j=n_2+1}^{\infty} \frac{C_W(j+1)\left(\Delta t_i\right)^{j+2}}{(j+2)!} \tag{49}$$

and employing equation (48) gives

$$\left\|\sum_{j=n_2+1}^{\infty} \frac{C_W(j+1)\left(\Delta t_i\right)^{j+2}}{(j+2)!}\right\| \leqq \sum_{j=n_2+1}^{\infty} \frac{\left\|C_W(j+1)\right\|\left(\Delta t_i\right)^{j+2}}{(j+2)!} \leqq \left\|Q_c\right\| \, \|B\|$$

$$\sum_{j=n_2+1}^{\infty} \frac{\|A\|^j 2^{j+1}\left(\Delta t_i\right)^{j+2}}{(j+2)!} < \left[\frac{2^{\left(n_2+2\right)}\|A\|^{\left(n_2+2\right)}\left(\Delta t_i\right)^{\left(n_2+3\right)}\left\|Q_c\right\| e^{2\|A\|\Delta t_i}}{\left(n_2+3\right)!}\right]\frac{\|B\|}{\|A\|} \tag{50}$$

Thus

$$\frac{W\left(\Delta t_i\right)}{2} = \sum_{j=1}^{n_2} \frac{C_W(j)\left(\Delta t_i\right)^{j+1}}{(j+1)!} + E_W\left(n_2\right) \tag{51}$$

where

$$\left\|E_W\left(n_2\right)\right\| \leqq E\left(n_2 + 1\right)\frac{\|B\|}{\|A\|} \tag{52}$$

for $\|A\| \neq 0$ and $n_2 \geqq 1$.

EVALUATION OF $R_d\left(\Delta t_i\right)$

The relation $R_d\left(\Delta t_i\right)$ can be written as

14

$$R_d\left(\Delta t_i\right) = R_c \, \Delta t_i + \tilde{R}_d\left(\Delta t_i\right) \tag{53}$$

where

$$\tilde{R}_d\left(\Delta t_i\right) = \int_0^{\Delta t_i} H'(\tau,0) \, Q_c \, H(\tau,0) \, d\tau \tag{54}$$

Define $L(\tau)$ as

$$L(\tau) = B'\left[\int_0^\tau e^{A'z} \, dz\right] Q_c \left[\int_0^\tau e^{Aw} \, dw\right] B \tag{55}$$

from which

$$\frac{dL(\tau)}{d\tau} = B'V(\tau) + V'(\tau) \, B \tag{56}$$

with $V(\tau)$ defined from equation (34). Recursively,

$$\frac{d^{j+1}}{d\tau^{j+1}} L(\tau) = B' \frac{d^j V}{d\tau^j}(\tau) + \frac{d^j V'}{d\tau^j}(\tau) \, B \qquad (j = 0, 1, \ldots) \tag{57}$$

The function $L(\tau)$ may alternately be expressed as

$$L(\tau) = \sum_{j=0}^\infty \left. \frac{d^j L(\tau)}{d\tau^j}\right|_{\tau=0} \frac{\tau^j}{j!} \tag{58}$$

With

$$C_R(j) = \left. \frac{d^j L(\tau)}{d\tau^j}\right|_{\tau=0} \tag{59}$$

15

equation (57) gives

$$C_R(j + 1) = B'C_W(j) + C'_W(j) B \tag{60}$$

with

$$C_R(0) = C_R(1) = 0$$

Then

$$\tilde{R}_d\left(\Delta t_i\right) = \int_0^{\Delta t_i} L(\tau) \, d\tau = \sum_{j=2}^{\infty} C_R(j) \frac{\left(\Delta t_i\right)^{j+1}}{(j+1)!} \tag{61}$$

For computational purposes, let

$$G(j) = \frac{C_R(j)\left(\Delta t_i\right)^{j+1}}{(j+1)!} \tag{62}$$

with

$$G(j + 1) = \left(\frac{B' \, \Delta t_i}{j+2}\right) D(j) + D'(j)\left(\frac{B \, \Delta t_i}{j+2}\right) \qquad (j = 0, 1, \ldots) \tag{63}$$

from which

$$\tilde{R}_d\left(\Delta t_i\right) = G(2) + G(3) + \ldots \tag{64}$$

By following the same procedure as in the previous sections, an upper bound on the error in truncating equation (61) after $j = n_3 \geqq 2$ terms can also be found as follows:

$$\tilde{R}_d\left(\Delta t_i\right) = \sum_{j=0}^{n_3} C_R(j + 2) \frac{\left(\Delta t_i\right)^{j+3}}{(j+3)!} + E_{\tilde{R}}\left(n_3\right) \tag{65}$$

16

$$\left\|C_R(j)\right\| \leq \left\|Q_c\right\| \left\|B\right\|^2 \left\|A\right\|^{j-2}\left(2^j - 2\right) \tag{66}$$

and

$$\left\|E_{\widetilde{R}}(n_3)\right\| < E(n_3 + 2)\left(\frac{\|B\|}{\|A\|}\right)^2 \tag{67}$$

if $A \neq 0$.

SUMMARY OF RESULTS

In the previous sections it has been established, for $\|A\| \neq 0$, that

$$Q_d\left(\Delta t_i\right) = \int_0^{\Delta t_i} e^{A'\tau} Q_c \, e^{A\tau} \, d\tau = Y(0) + Y(1) + \ldots + Y\left(n_1\right) + E_Q\left(n_1\right)$$

$$Y(j + 1) = \left(\frac{A' \, \Delta t_i}{j + 2}\right) Y(j) + Y(j)\left(\frac{A \, \Delta t_i}{j + 2}\right) \qquad \left(Y(0) = Q_c \, \Delta t_i\right)$$

$$\left\|E_Q\left(n_1\right)\right\| < E\left(n_1\right) = \frac{2^{\left(n_1+1\right)}\|A\|^{\left(n_1+1\right)}\left(\Delta t_i\right)^{\left(n_1+2\right)}\|Q_c\| \, e^{2\|A\|\Delta t_i}}{\left(n_1 + 2\right)!}$$

$$\frac{W\left(\Delta t_i\right)}{2} = \int_0^{\Delta t_i} e^{A'\tau} Q_c \int_0^\tau e^{Az} B \, dz \, d\tau = D(1) + D(2) + \ldots + D\left(n_2\right) + E_W\left(n_2\right)$$

$$D(j + 1) = \left(\frac{A' \, \Delta t_i}{j + 2}\right) D(j) + Y(j)\left(\frac{B \, \Delta t_i}{j + 2}\right) \qquad \left(D(0) = 0; \quad Y(0) = Q_c \, \Delta t_i\right)$$

$$\left\|E_W(n_2)\right\| < E\left(n_2 + 1\right)\frac{\|B\|}{\|A\|}$$

17

$$R_d\left(\Delta t_i\right) = R_c\, \Delta t_i + \tilde{R}_d\left(\Delta t_i\right)$$

$$\tilde{R}_d\left(\Delta t_i\right) = B' \int_0^{\Delta t_i} \int_0^{\tau} e^{A'z}\, dz\, Q_c \int_0^{\tau} e^{Aw}\, dw\, d\tau\, B$$

$$= G(2) + G(3) + \ldots + G\left(n_3\right) + E_{\tilde{R}}\left(n_3\right)$$

$$G(j+1) = \left(\frac{B'\, \Delta t_i}{j+2}\right) D(j) + D'(j)\left(\frac{B\, \Delta t_i}{j+2}\right) \qquad \left(D(1) = \frac{Q_c\, B\, \Delta t_i^2}{2}\right)$$

$$\left\|E_{\tilde{R}}\left(n_3\right)\right\| < E\left(n_3 + 2\right)\left(\frac{\|B\|}{\|A\|}\right)^2$$

The absolute error bounds for $E_Q\left(n_1\right)$, $E_W\left(n_2\right)$, and $E_{\tilde{R}}\left(n_3\right)$ can, in each case, be written in terms of the error function E. Relative accuracy requirements, however, generally require different values for n_1, n_2, and n_3.

For the trivial case in which $\|A\| = 0$,

$$Q_d\left(\Delta t_i\right) = Q_c\, \Delta t_i \tag{68}$$

$$W\left(\Delta t_i\right) = Q_c\, B\left(\Delta t_i\right)^2 \tag{69}$$

$$R_d\left(\Delta t_i\right) = R_c\, \Delta t_i + B'Q_c\, B\, \frac{\left(\Delta t_i\right)^3}{3} \tag{70}$$

COMPUTER IMPLEMENTATION

The infinite series representations summarized in the previous section, when numerically evaluated, yield $Q_d\left(\Delta t_i\right)$, $W\left(\Delta t_i\right)$, and $\tilde{R}_d\left(\Delta t_i\right)$ for general A, B, Q_c, and R_c matrices and sampling intervals Δt_i. In practice, numerical difficulties can occur unless the matrix A is preconditioned to avoid large numbers of terms in the series solutions and excessive roundoff error. One preconditioning approach and its implementation are presented in this section.

18

If the matrix A and sample time Δt_i are given, find an integer $K \geqq 0$ such that

$$\frac{\Delta t_i}{2^K} < \frac{1}{||A||} \tag{71}$$

By defining

$$\Delta = \frac{\Delta t_i}{2^K} \tag{72}$$

inequality (71) gives

$$\left|\left|A\Delta\right|\right| < 1 \tag{73}$$

whereby the eigenvalues of $A\Delta$ are within the unit circle in the complex plane and

$$\lim_{i \to \infty} (A\Delta)^i = 0 \tag{74}$$

Next, replace Δt_i by Δ and compute the corresponding $Q_d(\Delta)$, $W(\Delta)$, and $\tilde{R}_d(\Delta)$ matrices by the series solutions. Convergence should be rapid because of conditions (73) and (74). If $K > 0$, the required solutions for Δt_i may be constructed from $Q_d(\Delta)$, $W(\Delta)$, and $\tilde{R}_d(\Delta)$ by applying the equations derived in appendix A as follows. In equations (A5), (A7), (A8), and (A9) replace Δ by $2^j\Delta$ to give

$$Q_d\left(2^{j+1}\Delta\right) = Q_d\left(2^j\Delta\right) + e^{A'2^j\Delta} Q_d\left(2^j\Delta\right) e^{A2^j\Delta} \tag{75}$$

$$W\left(2^{j+1}\Delta\right) = \left(I + e^{A'2^j\Delta}\right) W\left(2^j\Delta\right) + 2e^{A'2^j\Delta} Q_d\left(2^j\Delta\right) H\left(2^j\Delta, 0\right) \tag{76}$$

$$\tilde{R}_d\left(2^{j+1}\Delta\right) = 2\tilde{R}_d\left(2^j\Delta\right) + \frac{W'\left(2^j\Delta\right)}{2} H\left(2^j\Delta, 0\right) + H'\left(2^j\Delta, 0\right)\frac{W\left(2^j\Delta\right)}{2}$$

$$+ H'\left(2^j\Delta, 0\right) Q_d\left(2^j\Delta\right) H\left(2^j\Delta, 0\right) \tag{77}$$

$$H\left(2^{j+1}\Delta, 0\right) = \left(I + e^{A2^j\Delta}\right) H\left(2^j\Delta, 0\right) \tag{78}$$

$$e^{A2^{j+1}\Delta} = e^{A2^j\Delta} e^{A2^j\Delta} \tag{79}$$

Recursively evaluating equations (75) to (79) from $j = 0$ to $j = K - 1$ yields $Q_d(\Delta t_i)$, $W(\Delta t_i)$, and $\tilde{R}_d(\Delta t_i)$ at the final stage. Algorithms for computing $e^{A\Delta}$ and $H(\Delta, 0)$ to initialize the recursive process are given by Källström. (See ref. 7.)

Software for the theory presented in this paper is available in the ORACLS program. (See refs. 8 and 9.)

<div align="center">AN EXAMPLE COMPUTATION</div>

In this section a particular set of $(A, B, Q_c, \Delta t_i)$ is chosen and results from the algorithms presented in the foregoing sections are illustrated. The computation was performed by using the subroutine SAMPL of the ORACLS program (ref. 9) on a CDC 6600 digital computer in single precision. The SAMPL subroutine employs the method of computer implementation described in the preceding section. Numerically, convergence was assumed to have occurred in the Q_d, W, and $\tilde{R}_d$ series when the improvement in the element of largest magnitude (measured relatively if the magnitude was less than unity, and absolutely otherwise) of each of the matrices was past the eighth significant digit. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\Delta t_i = 1/2$$

Taking

$$\left\|C\right\| = \max_{j} \sum_{i=1}^{m} \left\|c_{ij}\right\| \qquad\qquad (1 \leqq j \leqq n)$$

for some given $m \times n$ matrix C with elements c_{ij} gives

$$\left\|A\right\| = \left\|A^{\cdot}\right\| = 4$$

and

$$\left\|B\right\| = \left\|B^{\cdot}\right\| = 1$$

The condition

$$\frac{\Delta t_i}{2^K} < \frac{1}{\left\|A\right\|}$$

gives $K = 2$ and $\Delta = 1/8$. The results of the series computations are

$$Q_d(1/8) = \begin{bmatrix} 0.166270 & 0.0242575 & 0.0242575 \\ 0.0242575 & 0.166270 & 0.0242575 \\ 0.0242575 & 0.0242575 & 0.166270 \end{bmatrix}$$

$$\frac{W}{2}(1/8) = \begin{bmatrix} 0.0102932 & 0.00142899 \\ 0.00142899 & 0.0102932 \\ 0.00142899 & 0.00142899 \end{bmatrix}$$

$$\tilde{R}_d(1/8) = \begin{bmatrix} 0.000798547 & 0.0000827449 \\ 0.0000827449 & 0.000798547 \end{bmatrix}$$

with truncation error bounds

$$\left\|E_Q(n_1)\right\| < E(n_1) = 7.09 \times 10^{-10} \qquad\qquad \left(n_1 = 10\right)$$

21

$$\left\|E_W(n_2)\right\| < E(n_2 + 1) \frac{\|B\|}{\|A\|} = 1.77 \times 10^{-10} \qquad \left(n_2 = 9\right)$$

$$\left\|E_{\widetilde{R}}(n_3)\right\| < E(n_3 + 2)\left(\frac{\|B\|}{\|A\|}\right)^2 = 4.43 \times 10^{-11} \qquad \left(n_3 = 8\right)$$

Relative accuracies are

$$\frac{\left\|E_Q(n_1)\right\|}{\left\|Q_d(1/8)\right\|} < 3.30 \times 10^{-9}$$

$$\frac{\left\|E_W(n_2)\right\|}{\left\|\frac{W}{2}(1/8)\right\|} < 1.35 \times 10^{-8}$$

$$\frac{\left\|E_{\widetilde{R}}(n_3)\right\|}{\left\|\widetilde{R}_d(1/8)\right\|} < 5.03 \times 10^{-8}$$

Since $K = 2$, two passes through equations (75) to (77) were required which generated the sequence

$$Q_d(1/4) = \begin{bmatrix} 0.482451 & 0.158090 & 0.158090 \\ 0.158090 & 0.482451 & 0.158090 \\ 0.158090 & 0.158090 & 0.482451 \end{bmatrix}$$

$$\frac{W}{2}(1/4) = \begin{bmatrix} 0.0576453 & 0.0173101 \\ 0.0173101 & 0.0576453 \\ 0.0173101 & 0.0173101 \end{bmatrix}$$

$$\widetilde{R}_d(1/4) = \begin{bmatrix} 0.00815427 & 0.00184446 \\ 0.00184446 & 0.00815427 \end{bmatrix}$$

$$Q_d(1/2) = \begin{bmatrix} 2.80602 & 1.94688 & 1.94688 \\ 1.94688 & 2.80602 & 1.94688 \\ 1.94688 & 1.94688 & 2.80602 \end{bmatrix}$$

22

$$\frac{W}{2}(1/2) = \begin{bmatrix} 0.565488 & 0.355069 \\ 0.355069 & 0.565488 \\ 0.355069 & 0.355069 \end{bmatrix}$$

$$\tilde{R}_d(1/2) = \begin{bmatrix} 0.124575 & 0.0628764 \\ 0.0628764 & 0.124575 \end{bmatrix}$$

$$R_d(1/2) = \begin{bmatrix} 0.624575 & 0.0628764 \\ 0.0628764 & 0.624575 \end{bmatrix}$$

These final results agree with the closed form solutions shown in appendix B.

CONCLUDING REMARKS

A computational procedure for generating the weighting matrices needed in the time-invariant optimal linear sampled-data regulator problem has been presented. This procedure makes use of an analytical property of the defining equations to produce general purpose numerically attractive infinite series expansions which can be easily summed on a digital computer. Error bounds for truncating the series expansions after a finite number of terms were derived and gave good agreement with intended accuracy for the numerical example considered. It is felt that the results of this paper eliminate a basic and major difficulty in optimal sampled-data regulator methodology and open the way to a wider application of the theory.

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APPENDIX A

EVALUATION OF SAMPLED-DATA WEIGHTING MATRICES FOR

DOUBLED VALUES OF SAMPLING INTERVAL

For the purposes of this appendix, let the sampling interval be denoted by Δ and the weighting matrices defined by equations (11), (12), and (54) be denoted by

$$Q_d(\Delta) = \int_0^\Delta e^{A'\tau} Q_c e^{A\tau} d\tau \tag{A1}$$

$$\frac{W(\Delta)}{2} = \int_0^\Delta e^{A'\tau} Q_c H(\tau,0) d\tau \tag{A2}$$

$$\tilde{R}_d(\Delta) = \int_0^\Delta H'(\tau,0) Q_c H(\tau,0) d\tau \tag{A3}$$

with, as before,

$$H(\tau,0) = \int_0^\tau e^{Az} dz\, B \tag{A4}$$

Here equations for $Q_d(2\Delta)$, $W(2\Delta)$, and $\tilde{R}_d(2\Delta)$ are derived in terms of $Q_d(\Delta)$, $W(\Delta)$, $\tilde{R}_d(\Delta)$, $e^{A\Delta}$, and $H(\Delta,0)$. These equations are used in the body of the paper in the section dealing with a computer implementation of the algorithms for $Q_d\left(\Delta t_i\right)$, $W\left(\Delta t_i\right)$, and $R_d\left(\Delta t_i\right)$.

For Q_d,

$$Q_d(2\Delta) = \int_0^{2\Delta} e^{A'\tau} Q_c e^{A\tau} d\tau = Q_d(\Delta) + \int_0^\Delta e^{A'(\tau+\Delta)} Q_c e^{A(\tau+\Delta)} d\tau$$

$$= Q_d(\Delta) + e^{A'\Delta} Q_d(\Delta) e^{A\Delta} \tag{A5}$$

Additionally,

24

$$H(\tau+\Delta,0) = \int_0^{\tau+\Delta} e^{Az} \, B \, dz = H(\Delta,0) + \int_0^{\tau} e^{A(z+\Delta)} \, B \, dz$$

$$= H(\Delta,0) + e^{A\Delta} \, H(\tau,0)$$

from which

$$H(\tau+\Delta,0) = H(\Delta,0) + e^{A\Delta} \, H(\tau,0) = H(\tau,0) + e^{A\tau} \, H(\Delta,0) \tag{A6}$$

and

$$H(2\Delta,0) = \left(I + e^{A\Delta}\right) H(\Delta,0) \tag{A7}$$

For W,

$$\frac{W(2\Delta)}{2} = \int_0^{2\Delta} e^{A'\tau} Q_c \, H(\tau,0) \, d\tau = \frac{W(\Delta)}{2} + \int_0^{\Delta} e^{A'(\tau+\Delta)} Q_c \, H(\tau+\Delta,0) \, d\tau$$

$$= \frac{W(\Delta)}{2} + e^{A'\Delta} \left[\int_0^{\Delta} e^{A'\tau} Q_c \, H(\tau,0) \, d\tau + \int_0^{\Delta} e^{A'\tau} Q_c \, e^{A\tau} \, d\tau \, H(\Delta,0) \right]$$

whereby

$$W(2\Delta) = \left(I + e^{A'\Delta}\right) W(\Delta) + 2e^{A'\Delta} Q_d(\Delta) \, H(\Delta,0) \tag{A8}$$

Finally for $\tilde{R}_d$,

$$\tilde{R}_d(2\Delta) = \int_0^{2\Delta} H'(\tau,0) Q_c \, H(\tau,0) \, d\tau = \tilde{R}_d(\Delta) + \int_0^{\Delta} H'(\tau+\Delta,0) Q_c \, H(\tau+\Delta,0) \, d\tau$$

$$= \tilde{R}_d(\Delta) + \int_0^{\Delta} H'(\tau,0) Q_c \, H(\tau,0) \, d\tau + \int_0^{\Delta} H'(\tau,0) Q_c \, e^{A\tau} \, d\tau \, H(\Delta,0)$$

$$+ \; H'(\Delta,0) \int_0^\Delta e^{A'\tau} \, Q_c \, H(\tau,0) \; d\tau + H'(\Delta,0) \int_0^\Delta e^{A'\tau} \, Q_c \, e^{A\tau} \, d\tau \, H(\Delta,0)$$

or

$$\widetilde{R}_d(2\Delta) = 2\widetilde{R}_d(\Delta) + \frac{W'(\Delta)}{2} \, H(\Delta,0) + H'(\Delta,0) \, \frac{W(\Delta)}{2} + H'(\Delta,0) \, Q_d(\Delta) \, H(\Delta,0) \quad \text{(A9)}$$

A CLOSED FORM SOLUTION FOR COMPUTATIONAL EXAMPLE

A closed form solution is given for the numerical example presented in the body of this paper. These results are useful in determining the validity of the numerical computations.

For

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$e^{At} = e^t I + \frac{1}{3}\left(e^{4t} - e^t\right) F \tag{B1}$$

where I is a 3×3 identity matrix and

$$F = A - I \tag{B2}$$

With equation (B1),

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$R_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

equations (10), (11), and (12) yield

$$Q_d\left(\Delta t_i\right) = \begin{bmatrix} f_1 + f_2 & f_2 & f_2 \\ f_2 & f_1 + f_2 & f_2 \\ f_2 & f_2 & f_1 + f_2 \end{bmatrix}$$

$$W\left(\Delta t_i\right) = \begin{bmatrix} f_3 + f_4 & f_4 \\ f_4 & f_3 + f_4 \\ f_4 & f_4 \end{bmatrix}$$

$$R_d\left(\Delta t_i\right) = \begin{bmatrix} f_5 + f_6 & f_6 \\ f_6 & f_5 + f_6 \end{bmatrix}$$

where

$$f_1 = \frac{1}{2}\left(e^{2\Delta t_i} - 1\right) \tag{B3}$$

$$f_2 = \frac{1}{24}\left(e^{8\Delta t_i} - 1\right) - \frac{1}{6}\left(e^{2\Delta t_i} - 1\right) \tag{B4}$$

$$f_3 = \left(e^{2\Delta t_i} - 1\right) - 2\left(e^{\Delta t_i} - 1\right) \tag{B5}$$

$$f_4 = \frac{1}{48}\left(e^{8\Delta t_i} - 1\right) - \frac{1}{24}\left(e^{4\Delta t_i} - 1\right) - \frac{1}{3}\left(e^{2\Delta t_i} - 1\right) + \frac{2}{3}\left(e^{\Delta t_i} - 1\right) \tag{B6}$$

$$f_5 = \frac{1}{2}\left(e^{2\Delta t_i} - 1\right) - 2\left(e^{\Delta t_i} - 1\right) + 2\,\Delta t_i \tag{B7}$$

$$f_6 = \frac{\left(e^{8\Delta t_i} - 1\right)}{384} - \frac{1}{96}\left(e^{4\Delta t_i} - 1\right) - \frac{1}{6}\left(e^{2\Delta t_i} - 1\right) + \frac{2}{3}\left(e^{\Delta t_i} - 1\right) - \frac{5}{16}\Delta t_i \qquad (B8)$$

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